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СЫЗЫКТУУ ЭМЕС ГИПЕРБОЛИКАЛЫК ТЕҢДЕМЕЛЕРДИН КЕЧИКТИРҮҮ БОЮНЧА ЧЕКТЕЛГЕН ЧЫГАРЫЛЫШТАРЫ

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ОГРАНИЧЕННЫЕ РЕШЕНИЯ ЗАДЕРЖКИ НЕЛИНЕЙНЫХ ГИПЕРБОЛИЧЕСКИХ УРАВНЕНИЙ

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BOUNDED SOLUTIONS OF DELAY NONLINEAR HYPERBOLIC EQUATIONS

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H гильберт мейкиндигинде өзүнө-өзү түйүндөш оң белгилүү A оператору менен сызыктуу эмес дифференциалдык теңдемелердин кечиктирүү үчүн баштапкы маселени карап көрөлү.

$$\begin{cases} \dfrac{d^2u}{dt^2} + Au(t) = f\left(u(t), u(t-w)\right),\ t > 0, \\ \qquad u(t) = \varphi(t),\ -w \le t \le 0. \end{cases}$$

Бул маселенин чектелген чыгарылышынын жалгыздыгы жана жашашы жөнүндө теорема убактылуу кечиктирүү менен сызыктуу эмес гиперболикалык теңдемелер үчүн тургузулган. Убактылуу кечиктирүү менен төрт түрдүү сызыктуу эмес жекече туундудагы теңдемелер үчүн негизги теореманын колдонулушу көрсөтүлгөн.

***Негизги сөздөр**: чектелген чыгарылыш, сызыктуу эмес гиперболикалык теңдеме, чыгарылыштын жалгыздыгы жана жашашы.*

Рассмотрим начальную задачу

$$\begin{cases} \dfrac{d^2u}{dt^2} + Au(t) = f\left(u(t), u(t-w)\right),\ t > 0, \\ \qquad u(t) = \varphi(t),\ -w \le t \le 0 \end{cases}$$

для задержки нелинейного дифференциального уравнения в гильбертовом пространстве H с самосопряженным положительно определенным оператором A. Теорема о существовании и единственности ограниченного решения этой задачи установлена для нелинейного гиперболического уравнения с временной задержкой. Показано применение основной теоремы для четырех различных нелинейных уравнений с частными производными с временной задержкой.

***Ключевые слова**: ограниченные решения, нелинейное гиперболическое уравнение, единственность и существование решения.*

We consider the initial value problem

$$\begin{cases} \dfrac{d^2u}{dt^2} + Au(t) = f\left(u(t), u(t-w)\right),\ t > 0, \\ \qquad u(t) = \varphi(t),\ -w \le t \le 0 \end{cases}$$

for a delay nonlinear differential equation in a Hilbert space H with the self adjoint positive definite operator A. Theorem on the existence and uniqueness of a bounded solution of this problem is established for a nonlinear hyperbolic equation with time delay. The application of the main theorem for four different nonlinear partial differential equations with time delay is shown.

***Key words**: a bounded solution, a nonlinear hyperbolic equationб existence and uniqueness of a bounded solution.*

1. Introduction

Delay differential equations are used to model biological, physical, and sociological processes, as well as naturally occurring oscillatory systems (see, for examples, [1]-[4]). It is known that, in delay differential equations, the presence of the delay term causes the difficulties in analysis of differential equations. Lu [5], studies monotone iterative schemes for finite-difference solutions of reaction-diffusion systems with time delays and gives modified iterative schemes by combing the method of upper-lower solutions and the Jacobi method or the Gauss-Seidel method. Ashyralyev and Sobolevskii [6], consider the initial-value problem for linear delay partial differential equations of the parabolic type and give a sufficient condition for the stability of the solution of this initial-value problem. They obtain the stability estimates in HÖlder norms for the solutions of the problem.

Ashyralyev and Agirseven [7]-[13], investigated several types of initial and boundary value problems for linear delay parabolic equations. They give theorems on stability and convergence.

Moreover, Ashyralyev, Agirseven and Ceylan [14], interested in finding sufficient conditions for the existence of a unique bounded solution of the initial value problem for

$$\begin{cases} \dfrac{du}{dt} + Au(t) = f\left(u(t), u(t-w)\right),\ t > 0, \\ \qquad u(t) = \varphi(t),\ -w \le t \le 0 \end{cases} \tag{1}$$

the differential equation in a Banach space E with the positive operator A with dense domain $D(A)$. The main theorem on the existence and uniqueness of a bounded solution of problem (1) was established for a nonlinear evolutionary equation with time delay. The application of the main theorem for four different nonlinear partial differential equations with time delay was shown. Numerical results were shown.

It is known that various initial-boundary value problems for evolutionary nonlinear delay partial differential equations can be reduced to the initial value problem for the differential equation

$$\begin{cases} \dfrac{d^2u}{dt^2} + Au(t) = f\left(u(t), u(t-w)\right),\ t > 0, \\ \qquad u(t) = \varphi(t),\ -w \le t \le 0 \end{cases} \tag{2}$$

in a Hilbert space H with the self adjoint positive definite operator A with dense domain $D(A)$. Let $\{c(t),\ t \ge 0\}$ be a strongly continuous cosine operator-function defined by the formula

$$c(t) = \frac{e^{itA^{1/2}} + e^{-itA^{1/2}}}{2}.$$

Then, from the definition of the sine operator-function $s(t)$

$$s(t) = \int_0^t c(s)u\,ds$$

it follows that

$$s(t) = A^{-1/2}\frac{e^{itA^{1/2}} - e^{-itA^{1/2}}}{2i}.$$

The following estimates hold:

$$\|c(t)\|_{H\to H} \le 1, \left\|A^{1/2}s(t)\right\|_{H\to H} \le 1,\ t > 0. \tag{3}$$

A function $u(t)$ is called a solution of problem (2) if the following conditions are satisfied:

1. $u(t)$ is twice continuously differentiable on the interval $[-\omega,\infty)$. The derivative at the endpoint $t=-\omega$ is understood as the appropriate unilateral derivative.
2. The element $u(t)$ belongs to $D(A)$ for all $t\in[-\omega,\infty)$, and the function $Au(t)$ is continuous on the interval $[-\omega,\infty)$.
3. $u(t)$ satisfies the equation and the initial condition (2).

In this paper, we are interested in finding sufficient conditions for the existence of a unique bounded solution of problem (2). The main theorem on the existence and uniqueness of a bounded solution of problem (2) is established for a nonlinear evolutionary equation with time delay. The application of the main theorem for four different nonlinear partial differential equations with time delay is shown. In general, it is not able to get exact solution of nonlinear problems. Therefore, the first and second order of accuracy difference schemes for the solution of one dimensional nonlinear hyperbolic equation with time delay are presented. Numerical results are shown. Note that bounded solutions of nonlinear one dimensional parabolic and hyperbolic partial differential equations with time delay have been investigated in earlier papers [15]-[19]. The generality of the approach considered in this paper, however, allows for treating a wider class of multidimensional delay nonlinear differential equations.

2. Main Existence and Uniqueness Theorem

The method of proof is based on reducing problem (2) to an integral equation

$$u(t)=c(t-(n-1)w)u((n-1)w)+s(t-(n-1)w)\frac{du((n-1)w)}{dt}+$$
$$+\int_{(n-1)w}^{t} s(t-y)f(u(y),u(y-w))dy,$$
$$(n-1)w\le t\le nw,\ n=1,2,3,...,u(t)=\varphi(t),-w\le t\le 0$$

in $[0,\infty)\times H\times H$ and the use of successive approximations. The recursive formula for the solution of problem (2) is

$$u_i(t)=c(t-(n-1)w)u_i((n-1)w)+s(t-(n-1)w)\frac{du_i((n-1)w)}{dt}+$$
$$+\int_{(n-1)w}^{t} s(t-y)f(u_{i-1}(y),u(y-w))dy,$$
$$u_0(t)=c(t-(n-1)w)u_0((n-1)w)+s(t-(n-1)w)\frac{du_0((n-1)w)}{dt},$$
$$(n-1)w\le t\le nw,\ n=1,2,...,$$
$$i=1,2,...,u_i(t)=\varphi(t),\ -w\le t\le 0. \tag{4}$$

Theorem 2.1. Assume the following hypotheses:

For any $t,-w\le t\le 0,\ \varphi(t)\in D(A)$ and

$$\|\varphi(t)\|_H\le M,\|A^{-1/2}\varphi'(t)\|_H\le \tilde{M}. \tag{5}$$

The function $f:H\times H\to H$ be continuous and bounded function, that is

$$\|A^{-1/2}f(u,\upsilon)\|_H\le \bar{M} \tag{6}$$

in $H\times H$and Lipschitz condition holds uniformly with respect to z

$$\left\|A^{-1/2}\left(f(u,z)-f(\upsilon,z)\right)\right\|_H\le L\|u-\upsilon\|_H. \tag{7}$$

Here, $L,M,\tilde{M},\bar{M}$ are positive constants. Then there exists a unique solution to problem (2) which is bounded in $[0,\infty)\times H\times H$.

Proof. We consider the interval $0\le t\le w.$ Problem (2) becomes

$$\frac{d^2u}{dt^2}+Au(t)=f(u(t),\varphi(t-w)),\ u(0)=\varphi(0),\ u'(0)=\varphi'(0)$$

and it can be written in equivalent integral form

$$u(t)=c(t)\varphi(0)+s(t)\varphi'(0)+\int_0^t s(t-y)f(u(y),\varphi(y-w))dy. \tag{8}$$

According to the method of recursive approximation (4), we get

$$u_i(t)=c(t)\varphi(0)+s(t)\varphi'(0)+\int_0^t s(t-y)f(u_{i-1}(y),\varphi(y-w))dy,i=1,2,.... \tag{9}$$

Therefore,

$$u(t)=u_0(t)+\sum_{i=0}^{\infty}(u_{i+1}(t)-u_i(t)), \tag{10}$$

where

$$u_0(t)=c(t)\varphi(0)+s(t)\varphi'(0).$$

Applying estimates (3) and (5), we get

$$\|u_0(t)\|_E \le \|c(t)\|_{H\to H}\|\varphi(0)\|_H+\left\|A^{1/2}s(t)\right\|_{H\to H}\left\|A^{-1/2}\varphi'(0)\right\|_H \le M+\tilde{M}.$$

Applying formula (9) and estimates (3) and (6), we get

$$\|u_1(t)-u_0(t)\|_H \le \int_0^t \left\|A^{1/2}s(t-y)\right\|\left\|A^{-1/2}f(u_0(y),\varphi(y-w))\right\|_H dy \le \bar{M}t.$$

Using the triangle inequality, we get

$$\|u_1(t)\|_H \le M+\tilde{M}+\bar{M}t.$$

Applying formula (9) and estimates (7),(3) and (6), we get

$$\|u_2(t)-u_1(t)\|_H \le \int_0^t \left\|A^{1/2}s(t-y)\right\|\left\|A^{-1/2}\left[f(u_1(y),\varphi(y-w))-f(u_0(y),\varphi(y-w))\right]\right\|_H dy \le$$

$$\le L\int_0^t \|u_1(t)-u_0(t)\|_H\, dy \le L\bar{M}\int_0^t y\,dy=\frac{\bar{M}}{L}\frac{(Lt)^2}{2!}.$$

Then

$$\|u_2(t)\|_H \le M+\tilde{M}+\frac{\bar{M}}{L}\frac{Lt}{1!}+\frac{\bar{M}}{L}\frac{(Lt)^2}{2!}.$$

Let

$$\|u_n(t)-u_{n-1}(t)\|_H \le \frac{\bar{M}}{L}\frac{(Lt)^n}{n!}.$$

Then, we obtain

$$\|u_{n+1}(t)-u_n(t)\|_H \le \int_0^t \left\|A^{1/2}s(t-y)\right\|\left\|A^{-1/2}\left[f(u_n(y),\varphi(y-w))-f(u_{n-1}(y),\varphi(y-w))\right]\right\|_H dy \le$$

$$\le L\int_0^t \|u_n(t)-u_{n-1}(t)\|_H\, dy \le \int_0^t L\frac{\bar{M}}{L}\frac{(Ly)^n}{n!}dy=\frac{\bar{M}}{L}\frac{(Lt)^{n+1}}{(n+1)!}.$$

Therefore, for any $n, n\ge 1$, we have that

$$\|u_{n+1}(t)-u_n(t)\|_H \le \frac{\bar{M}}{L}\frac{(Lt)^{n+1}}{(n+1)!}$$

and

$$\|u_{n+1}(t)\|_H \le M+\tilde{M}+\frac{\bar{M}}{L}\frac{Lt}{1!}+\dots+\frac{\bar{M}}{L}\frac{(Lt)^{n+1}}{(n+1)!}$$

by mathematical induction. From that and formula (10) it follows that

$$\|u(t)\|_H \le \|u_0(t)\|_H + \sum_{i=0}^{\infty} \|u_{i+1}(t) - u_i(t)\|_H \le M + \tilde{M} + \sum_{i=0}^{\infty} \frac{\bar{M}}{L} \frac{(Lt)^{i+1}}{(i+1)!} \le$$

$$\le M + \tilde{M} + \frac{\bar{M}}{L} e^{Lt},\ 0 \le t \le w$$

which proves the existence of a bounded solution of Problem (2) in $[0,w]\times H\times H$. Now, we consider solution of Problem (2) in $w \le t \le 2w$. We note that $0 \le t-w \le w$. We denote that

$$\varphi_1(t) = u(t-w), w \le t \le 2w.$$

Replacing *t* and *t-w* and assuming that

$$\left\| A^{-1/2} f(u_0(y), \varphi(y-w)) \right\|_H \le \bar{M}_1$$

and

$$\|\varphi_1(t)\|_H \le M_1,\ \left\| A^{-1/2} \varphi_1'(t) \right\|_H \le \tilde{M}_1.$$

Therefore,

$$u_0(t) = c(t-w)\varphi_1(w) + s(t-w)\frac{d\varphi_1(w)}{dt},$$

$$u_i(t) = c(t-w)\varphi_1(w) + s(t-w)\frac{d\varphi_1(w)}{dt} +$$

$$+\int_w^t s(t-y) f(u_{i-1}(y), u_i(y-w))dy, i = 1,2,...,$$

In a similar manner, for any $n,\ n \ge 1$, we obtain that

$$\|u_{n+1}(t) - u_n(t)\|_H \le \frac{\bar{M}_1}{L} \frac{(L(t-w))^{n+1}}{(n+1)!}$$

and

$$\|u_{n+1}(t)\|_E \le M_1 + \tilde{M}_1 + \frac{\bar{M}_1}{L}\frac{Lt}{1!} + ... + \frac{\bar{M}_1}{L}\frac{(L(t-w))^{n+1}}{(n+1)!}.$$

From that it follows that

$$\|u(t)\|_H \le M_1 + \tilde{M}_1 + \frac{\bar{M}_1}{L} e^{L(t-nw)},\ w \le t \le 2w$$

which proves the existence of a bounded solution of Problem (2) in $[w,\ 2w]\times H \times H$.

In a similar manner, we can obtain that

$$\|u(t)\|_H \le M_n + \tilde{M}_n + \frac{\bar{M}_n}{L} e^{L(t-nw)},\ nw \le t \le (n+1)w,$$

where $M_n, \tilde{M}_n$ and $\bar{M}_n$ are bounded. This proves the existence of a bounded solution of Problem (2) in $[nw,\ (\text{n+1})w]\times H \times H$. In general, the function $u(t)$ constructed is a solution of Problem (2) which is bounded in $[0,\infty)\times H\times H$.

Now we will prove uniqueness of this solution of Problem (2). Assume that there is a bounded solution $\upsilon(t)$ of Problem (2) and $\upsilon(t) \ne u(t)$. We denote that $z(t) = \upsilon(t) - u(t)$. Therefore for $z(t)$, we have that

$$\begin{cases} \dfrac{d^2 z(t)}{dt^2} + Az(t) = f(\upsilon(t), \upsilon(t-w)) - f(u(t), u(t-w)), \ t > 0 \\ z(t) = 0, \ -w \le t \le 0. \end{cases}$$

We consider the interval $0 \le t \le w$. Since $v(t-w) = v(t-w) = \varphi(t-w)$, we have that

$$\begin{cases} \dfrac{d^2 z(t)}{dt^2} + Az(t) = f\left(v(t), \varphi(t-w)\right) - f\left(u(t), \varphi(t-w)\right), t > 0, \\ \qquad z(t) = 0, -w \le t \le 0. \end{cases}$$

Therefore,

$$z(t) = \int_0^t s(t-y)[f\left(v(y), \varphi(y-w)\right) - f\left(u(y), \varphi(y-w)\right)]ds.$$

Applying estimates (3) and (6), we get

$$\|z(t)\|_H \le \int_0^t \left\| A^{1/2} s(t-y) \right\| \left\| [-A^{1/2} f\left(v(y), \varphi(y-w)\right) - f\left(u(y), \varphi(y-w)\right)] \right\|_H dy$$

$$\le L \int_0^t \|v(y) - u(y)\|_H \, ds \le L \int_0^t \|z(y)\|_H \, dy$$

Using the integral inequality, we get

$$\|z(y)\|_H \le 0$$

From that it follows that $\|z(t)\|_H \le 0$ which proves the uniqueness of a bounded solution of Problem (2) in $[0, w] \times H \times H$. Applying same way and mathematical induction, we can prove the uniqueness of a bounded solution of Problem (2) in $[0, \infty) \times H \times H$.

Remark 2.1. Method of present paper also enables to prove, under certain assumptions, the existence of a unique bounded solution of the initial value problem for evolutionary nonlinear partial differencial equations

$$\begin{cases} \dfrac{d^2 u}{dt^2} + Au(t) = f(u(t), u([t]), t > 0, \\ \quad u(0) = \varphi(0), u'(0) = \varphi'(0). \end{cases} \tag{11}$$

In a Hilbert space H with the self adjoint positive operator A with dense domain D(A). Here [t] denotes the greatest-integer function.

3. Applications

First, we consider the initial-boundary value problem for one dimensional nonlinear delay differencial equations of hyperbolic type

$$\begin{cases} \dfrac{d^2 u(t,x)}{dt^2} - \left(a(x) u_x(t,x)\right)_x + \delta u(t,x) = f(x, u(t,x), u(t-w,x)), \\ \qquad 0 < t < \infty, x \in (0,l) \\ u(t,x) = \varphi(t,x), \varphi(t,0) = \varphi(t,l), \varphi_x(t,0) = \varphi_x(t,l), -w \le t \le 0, x \in [0,l], \\ \qquad u(t,0) = u(t,l), u_x(t,0) = u_x(t,l), -w \le t < \infty \end{cases} \tag{12}$$

Where $a(x), \varphi(t,x)$ are given sufficiently smooth functions and $\delta > 0$ is the sufficiently large number. We will assume that $a(x) \ge \mathrm{a} > 0$ and $a(l) = \mathrm{a}(0)$.

Theorem 3.1. Assume the following hypotheses

1. For any $t, -w \le t \le 0$

$$\|\varphi(t,.)\|_{L_2[0,l]} \le M, \|\varphi'(t,.)\|_{L_2[0,l]} \le \tilde{M}. \tag{13}$$

2. The function $f:(0,l)\times L_2[0,l]\times L_2[0,l]\to L_2[0,l]$ be continuous and bounded, that is

$$\|f(u,v)\|_{L_2[0,l]} \le \bar{M}. \tag{14}$$

and Lipschitz condition holds uniformly with respect to z

$$\|f(u,z)-f(v,z)\|_{L_2[0,l]} \le L\|u-v\|_{L_2[0,l]} \tag{15}$$

Here and in future, $L, M, \tilde{M}, \bar{M}$ are positive constants. Then there exists a unique solution to problem (12) which is bounded in $[0,\infty)\times L_2[0,l]\times L_2[0,l]$.

The proof of Theorem 3.1 is based on the abstract Theorem 2.1, on the self-adjointness and positivity in $L_2[0,l]$ of a differential operator A^x defined by the formula

$$A^x u = -\frac{d}{dx}\left(a(x)\frac{du}{dx}\right)+\delta u \tag{16}$$

With domain $D\left(A^x\right)=\left\{u\in W_2^2[0,l]: u(0)=u(l), u'(0)=u'(l)\right\}$ [20] and on the estimate

$$\|c\{t\}\|_{L_2[0,l]\to L_2[0,l]} \le 1, \left\|\left(A^x\right)^{\frac{1}{2}} s\{t\}\right\|_{L_2[0,l]\to L_2[0,l]} \le 1, t\ge 0 \tag{17}$$

Second, we consider the initial nonlocal boundary value problem for one dimensional non-linear delay differential equations of hyperbolic type with involution

$$\begin{cases} \dfrac{d^2u(t,x)}{dt^2}-\left(a(x)u_x(t,x)\right)_x-\beta\left(a(-x)u_x(t,-x)\right)_x+\delta u(t,x) \\ = f(x,u(t,x),u(t-w,x)),\ 0<t<\infty, x\in(-l,l), \\ \qquad u(t,x)=\varphi(t,x), \varphi(t,-l)=\varphi(t,l)=0, \\ \qquad -w\le t\le 0,\ x\in[-l,l], \\ \qquad u(t,-l)=u(t,l)=0, -w\le t<\infty, \end{cases} \tag{18}$$

Where $a(x), \varphi(t,x)$ are given sufficiently smooth functions and $\delta>0$ is the sufficiently large number. We will assume that $\mathrm{a}\ge a(x)=a(-x)\ge\delta>0, \delta-a|\beta|\ge 0$.

Theorem 3.2. Assume the following hypotheses:

1. <u>For any $t, -w\le t\le 0$</u>

$$\|\varphi(t,.)\|_{L_2[-l,l]} \le M, \|\varphi'(t,.)\|_{L_2[-l,l]} \le \tilde{M}.$$

2. <u>The function $f:(-l,l)\times L_2[-l,l]\times L_2[-l,l]\to L_2[-l,l]$ be continuous and bounded, that is</u>

$$\|f(u,v)\|_{L_2[-l,l]} \le \bar{M}.$$

and Lipschitz condition holds uniformly with respect to z

$$\|f(u,z)-f(v,z)\|_{L_2[-l,l]} \le L\|u-v\|_{L_2[-l,l]} .$$

Then there exists a unique solution to problem (18) which is bounded in $[0,\infty)\times L_2[-l,l]\times L_2[-l,l]$.

The proof of Theorem 3.2 is based on the abstract Theorem 2.1, on the self-adjointness and positivity in $L_2[-l,l]$ of a differential operator A^x defined by the formula

$$A^x v(x) = -\left(a(x)v_x(x)\right)_x - \beta\left(a(-x)v_x(-x)\right)_x + \delta v(x)$$

With domain $D\left(A^x\right) = \left\{u \in W_2^2[-l,l] : u(-l)=u(l)=0\right\}$ [21] and on the estimate

$$\|c\{t\}\|_{L_2[-l,l]\to L_2[-l,l]} \le 1, \left\|\left(A^x\right)^{\frac{1}{2}} s\{t\}\right\|_{L_2[-l,l]\to L_2[-l,l]} \le 1 , t \ge 0 .$$

Third, let $\Omega \subset R^n$ be a bounded open domain with smooth boundary $S, \bar{\Omega} = \Omega \cup S$. In $[0,\infty)\times\Omega$ we consider the initial boundary value problem for multidimensional nonlinear delay differential equations of hyperbolic type

$$\begin{cases} \dfrac{d^2u(t,x)}{dt^2} - \sum_{r=1}^{n}\left(a_r(x)u_{x^r}\right)x^r + \delta u(t,x) \\ = f(x,u(t,x),u(t-w,x)),\ 0<t<\infty, x=(x_1,...,x_n)\in\Omega, \\ u(t,x)=\varphi(t,x), -w\le t\le 0,\ x\in\bar{\Omega}, \\ u(t,x)=0,\ x\in S, 0\le t<\infty, \end{cases} \tag{19}$$

Where $a_r(x)$ and $\varphi(t,x)$ are given sufficiently smooth functions and $\delta > 0$ is the sufficiently large number. We will assume that $a_r(x) > 0$.

Theorem 3.3. Assume the following hypotheses:

1. For any $t, -w \le t \le 0$

$$\|\varphi(t,.)\|_{L_2[\bar{\Omega}]} \le M , \|\varphi'(t,.)\|_{L_2[\bar{\Omega}]} \le \tilde{M}.$$

2. The function $f : Q\times L_2\left[\bar{\Omega}\right]\times L_2\left[\bar{\Omega}\right] \to L_2\left[\bar{\Omega}\right]$ be continuous and bounded, that is

$$\|f(u,v)\|_{L_2[\bar{\Omega}]} \le \bar{M}.$$

and Lipschitz condition holds uniformly with respect to z

$$\|f(u,z)-f(v,z)\|_{L_2[\bar{\Omega}]} \le L\|u-v\|_{L_2[\bar{\Omega}]} .$$

Then there exists a unique solution to problem (19) which is bounded in $[0,\infty)\times L_2\left[\bar{\Omega}\right]\times L_2\left[\bar{\Omega}\right]$.

The proof of Theorem 3.3 is based on the abstract Theorem 2.1, on the self-adjointness and positivity in $L_2\left[\bar{\Omega}\right]$ of a differential operator A^x defined by the formula

$$A^x u(x) = -\sum_{r=1}^{n}\left(a_r(x)u_{x^r}\right)_{x^r} + \delta u(x) \tag{20}$$

With domain ([22]) $D\left(A^x\right)=\left\{u(x):u(x),u_{x_r}(x),(a_r(x)u_{x_r})_{x_r}\in L_2\left(\bar{\Omega}\right),1\le r\le n,u(x)=0,x\in S\right\}$ and on the estimate

$$\left\|c\{t\}\right\|_{L_2[\bar{\Omega}]\to L_2[\bar{\Omega}]}\le 1,\left\|\left(A^x\right)^{\frac{1}{2}}s\{t\}\right\|_{L_2[\bar{\Omega}]\to L_2[\bar{\Omega}]}\le 1\,,t\ge 0 \tag{21}$$

Fourth, in $[0,\infty)\times\Omega$ we consider the initial boundary value problem for multidimensional nonlinear delay differential equations of hyperbolic type

$$\begin{cases}\dfrac{d^2u(t,x)}{dt^2}-\displaystyle\sum_{r=1}^{n}\left(a_r(x)u_{x^r}\right)x^r+\delta u(t,x)\\ =f(x,u(t,x),u(t-w,x)),\ 0<t<\infty,x=(x_1,...,x_n)\in\Omega,\\ u(t,x)=\varphi(t,x),-w\le t\le 0,\ x\in\bar{\Omega},\\ \dfrac{\partial u}{\partial\vec{n}}(t,x)=0,\ x\in S,0\le t<\infty,\end{cases} \tag{22}$$

Where $a_r(x)$ and $\varphi(t,x)$ are given sufficiently smooth functions and $\delta>0$ is the sufficiently large number and $a_r(x)>0$. Here, $\vec{n}$ is the normal vector to Ω .

Theorem 3.4. Suppose that assumptions of Theorem 3.3 hold. Then there exists a unique solution to problem (22) which is bounded in $[0,\infty)\times L_2\left[\bar{\Omega}\right]\times L_2\left[\bar{\Omega}\right]$.

The proof of Theorem 3.4 is based on the abstract Theorem 2.1, on the self-adjointness and positivity in $L_2\left[\bar{\Omega}\right]$ of a differential operator A^x defined by the formula

$$A^xu(x)=-\sum_{r=1}^{n}\left(a_r(x)u_{x^r}\right)_{x^r}+\delta u(x)$$

With domain ([22])

$D\left(A^x\right)=\left\{u(x):u(x),u_{x_r}(x),(a_r(x)u_{x_r})_{x_r}\in L_2\left(\bar{\Omega}\right),1\le r\le n,\dfrac{\partial u}{\partial\vec{n}}=0,x\in S\right\}$ and on the estimate (21).

4. Numerical Results

In general, it is not possible to get exact solution of nonlinear problems. Therefore, the first and second order of accuracy difference schemes for the solution of one dimensional nonlinear hyperbolic equation with time delay are presented. Numerical results are provided. We consider the initial-boundary value problem

$$\begin{cases}\dfrac{d^2u(t,x)}{dt^2}-\dfrac{d^2u(t,x)}{dt^2}=f\left(x,u(t,x),u(t-1,x)\right),\\ f(x,u(t,x),u(t-1,x))=2e^{-t}\sin x\\ +\sin\left(u(t,x)\left[u(t-1,x)\cos x-\dfrac{\partial u(t-1,x)}{\partial x}\sin x\right]\right),0<t<\infty,0<x<\pi,\\ u(t,x)=e^{-t}\sin x,0\le x\le\pi,\ -1\le t\le 0,\\ u(t,0)=u(t,\pi)=0,\ t\ge 0\end{cases} \tag{23}$$

for the nonlinear delay hyperbolic differential equation. The exact solution of this test example is $u(t,x)=e^{-t}\sin x$.

We get the following iterative difference scheme of first order of accuracy in t for the approximate solution of the initial-boundary value problem (23)

$$\begin{cases} \dfrac{{}_m u_n^{k+1}-2\left(u_n^k\right)+u_n^{k-1}}{\tau}-\dfrac{{}_m u_{n+1}^{k+1}-2\left(u_n^{k+1}\right)+u_{n-1}^{k+1}}{h^2}=2e^{-t}\sin x_n \\ +\sin\left(({}_{m-1}u_n^k)\left(({}_m u_n^{k-N})\cos x_n-\dfrac{{}_m u_{n+1}^{k-N}-\left({}_m u_{n-1}^{k-N}\right)}{2h}\sin x_n\right)\right), \\ t_k=k\tau, x_n=nh, 1\le k<\infty, 1\le n\le M-1, \\ {}_m u_n^k=e^{-t_k}\sin x_n, x_n=nh, 0\le n\le M, t_k=k\tau, -N\le k\le 0, N_\tau=1, Mh=\pi, \\ {}_m u_0^k={}_m u_M^k=0, 0\le k<\infty, m=1,2,... \end{cases} \quad (24)$$

for the nonlinear delay hyperbolic equation. Here and in future m denotes the iteration index and an initial guess $0u_n^k, k\ge 1, 0\le n\le M$ is to be made. Applying the modified Gauss elimination method, equation (24) is solved. In computations the initial guess $0u_n^k$ is chosen and when the maximum error between two consecutive results of iterative difference scheme (24) becomes less than 10^{-8}, the iterative process is terminated.

Now, we get the following iterative difference scheme of second order of accuracy in t for the approximate solution of the initial-boundary value problem (23)

$$\begin{cases} \dfrac{{}_m u_n^{k+1}-2\left(u_n^k\right)+u_n^{k-1}}{\tau}-\dfrac{{}_m u_{n+1}^{k+1}-2\left(u_n^{k+1}\right)+u_{n-1}^{k+1}}{2h^2}-\dfrac{{}_m u_{n+1}^{k-1}-2\left(u_n^{k-1}\right)+u_{n-1}^{k-1}}{2h^2}=2e^{-t}\sin x_n \\ +\sin\left(\dfrac{1}{2}\left[({}_{m-1}u_n^{k+1})+({}_{m-1}u_n^{k-1})\right]\left(({}_m u_n^{k-N})\cos x_n-\dfrac{{}_m u_{n+1}^{k-N}-\left({}_m u_{n-1}^{k-N}\right)}{2h}\sin x_n\right)\right), \\ t_k=k\tau, x_n=nh, 1\le k<\infty, 1\le n\le M-1, \\ {}_m u_n^k=e^{-t_k}\sin x_n, x_n=nh, 0\le n\le M, t_k=k\tau, -N\le k\le 0, N_\tau=1, Mh=\pi, \\ {}_m u_0^k={}_m u_M^k=0, 0\le k<\infty, m=1,2,... \end{cases} \quad (25)$$

In Table 1 as we increase M and N values each time starting from $M=N=30$ by a factor of 2 the errors in the first order of accuracy difference scheme decrease approximately by a factor of $1/2$, the errors in the second order of accuracy difference scheme decrease approximately by a factor of $1/4$. The errors presented in the table indicate the stability of the difference schemes and the accuracy of the results. Thus, the second order of accuracy difference scheme increases faster than the first order of accuracy difference scheme.

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