ТОПОЛОГИЯ И ГЕОМЕТРИЯ

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COMPACTIFICATION OF UNIFORMILY CONTINOUS MAPPINS

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In this paper the notion of compactification of a uniformly continuous mapping is introduced and some of properties are established.

The notion of compactification of continuous mappings has been introduced and studied in [3], [4]. A wider study of compactification of continuous mappings has been done in the paper of Pasynkov [1] and papers [2], [3], [4], [5], [6].

All considered uniform spaces are assumed to be separated and given in terms of coverings, mappings are uniformly continuous and topological spaces are Tychonoff.

Definition 1. *Let $f:(\mathrm{X},\mathrm{U})\to(\mathrm{Y},\mathrm{V})$ be a uniformly continuous mapping. A mapping $cf:(cX,cY)\to(Y,V)$ is called compactification or uniformly perfect extension of the mapping f if following conditions are met: 1) $X\subseteq cX$; 2) $[X]_{cX}\subseteq cX$; 3) $cf\big|_X=f$; 4) cf is a uniformly perfect map.*

For two compactification $c_1f:(c_1X,c_1U)\to(Y,\mathrm{V})$ and $c_2f:(c_2X,c_2U)\to(Y,\mathrm{V})$ of a mapping $f:(X,U)\to(Y,\mathrm{V})$, as usually we set $c_2f\geq c_1f$, if there is a uniformly continuous mapping $\varphi:(c_2X,c_2U)\to(c_1X,c_1U)$ such that $\mathrm{c}_2\,f=c_1f*\varphi$ and φ is an identity mapping on X.

The notions of uniformly perfect and complete mappings are introduced and investigated by the author in [7], [8], [9], [10].

Here are some necessary notions and statements.

Let $f:(X,\mathrm{U})\to(\mathrm{Y},\mathrm{V})$ be a uniformly continuous mapping. A uniformity $B\subseteq U$ is called a base of the mapping f, if for any cover $\alpha\subseteq U$ there are a cover $\beta\in V$ and a cover $\gamma=B$ such that the cover $f^{-1}\beta\wedge\gamma=\{f^{-1}B\cap\Gamma:B\in\beta,\Gamma\in\gamma\}$ is inscribed in the cover α.

If B is a precompact uniformity, then the mapping f is called precompact [7],[10].

A mapping $f:(X,\mathrm{U})\to(\mathrm{Y},\mathrm{V})$ is called to be *uniformly perfect* [7], if it is precompact and perfect in the topological sense.

A mapping $f:(X,\mathrm{U})\to(\mathrm{Y},\mathrm{V})$ is called complete [9], [10] if every Cauchy filter F in (X,U) converges in (X,U) if its image fF converges in (Y,V).

Note that a mapping $f:(X,U)\to(Y,V)$ is uniformly perfect if and only if it is both precompact and complete [10].

A mapping $bf:(bX,bU)\to(Y,V)$ is called a completion of the mapping $f:(X,U)\to(Y,V)$ (see [9], [10]) if the following conditions are met:

1) (X,U) is dense subspace of the uniform space (bX,bU); 2) $f=bf\big|_x$; 3) the mapping bf is complete.

It should be noted that the completion of a precompact mapping is uniformly perfect [10].

Lemma 1. *Let $f:(X,U)\to(Y,V)$ be a uniformly continuous mapping. Then is a uniformity Up on X such that*

1) $U_p\subseteq U$

2) the mapping $f:(X,U)\to(Y,V)$ is precompact;

3) U_p generates the same topology as U.

Among such infirmities there is maximal uniformity.

Proof. Let U_c be precompact uniformity contained in the uniformity U and generating the same topology as does U [11]. We denote by U_p the uniformity on X, generated by the system of converings $U_c\wedge f^{-1}V\{\gamma\wedge f^{-1}\beta:\gamma\in U_c,\beta\in V\}$. By the construction $U_p\subseteq U$ and the mapping

$f:(X,U_p)\to(Y,V)$ is precompact. Condition 3) of Lemma 1 is obvious. We show the exictence of maximal uniformity U_s, turning the mapping f to become precompact. To this end it's enough to take a maximal precompact uniformity U_m contained in U. We denote through U_s the uniformity generated by the system f covers $U_m \wedge f^{-1}V$. Then the uniformity U_s is the required one.

Theorem 1. *Every uniformly continuous mapping* $f:(X,U)\to(Y,V)$ *has at least one compactification.*

Proof. by Lemma 1, there is a uniformity U_p on X contained in U and making the mapping $f:(X,U_p)\to(Y,V)$ to be precompact. Let a mapping $cf:(\mathrm{c}X,\mathrm{c}U)\to(Y,V)$ be complation of the mapping $f:(X,U_p)\to(Y,V)$, where $f=cf\big|_X;[X]_{cX}=cX$ and cf is complete.

In [10] it is proved that the completion of a precompact map is a uniformly perfect map. Then the mapping cf is a uniformly perfect extension of the mapping f, i.e., the mapping cf is a compactification of the mapping f. Theorem 1 is proved.

If U is a maximal uniformity of a Tychonoff space X [11], then the theorem implies the following corollary.

Corollary 1. *Every continuous mapping* $f:(X,U)\to(Y,V)$ *of a Tychonoff space* X *into a Tychonoff space* Y *has at least one compactification.*

This corollary in the class of spaces wider than the class of Tychonoff spaces is proved in [1], [2].

Theorem 2. *Every uniformly continuous mapping* $f:(X,U)\to(Y,V)$ *has maximal compactification.*

Proof. By Lemma 1 , there is a maximum uniformity $U_s\subseteq U$, making the mapping $f:(X,U_s)\to(Y,V)$ precompact. Let sf: $(\mathrm{s}X,\mathrm{s}U)\to(Y,V)$ be the completion of the mapping $f:(X,U_s)\to(Y,V)$. Then, as mentioned above, sf is uniformly perfect [10], i.e. sf is a compactification of the mapping $f:(X,U)\to(Y,V)$. We shall show that the compactification sf is the maximal compactification of the mapping f.

Let $cf:(\mathrm{c}X,\mathrm{c}U)\to(Y,V)$ be an arbitrary compactification of the mapping $f:(X,U)\to(Y,V)$. We shall show that there exists a uniformly continuous mapping $\varphi:(\mathrm{s}X,\mathrm{s}U)\to(\mathrm{s}Y,\mathrm{s}V)$ such that $sf=cf\varphi$.

Actually, let $i_X:X\to X$ be an identity mapping. Let U_s Us be a restriction of the uniformity ${}_sU$ on X and U_c be a restriction of the uniformity ${}_cU$ on X. Then $U_c\subseteq U_s\subseteq U$, and identity mapping $i_X:\left(X,\ U_s\right)\ \to\ \left(Y,\ U_c\right)$ is uniformly continuous. We denote by φ the extension of i_X onto $\left(sX,\ sU\right)$ and $\left(cX,\ cU\right)$. It is easy to see that such an extension exists, it is uniformly continuous and $sf=cf\cdot\varphi$. Theorem 2 is proved.

Maximal compactification sf of the mapping f is called Samuel compactification of the mapping $f:(X,U_s)\to(Y,V)$.

If U is the maximal uniformity of a Tychonoff space X, then Theorem 2 implies the following corollary.

Corollary 2. *Among all compactifications of a continuous mapping* $f:X\to Y$ *there is a maximal compactification.*

This corollary in the class of spaces is proved in [1], [2].

Let $f:(X,U)\to(Y,V)$ be uniformly continuous. Through U_s and V_s we denote precompact maximal uniformities, contained in the uniformities U and V respectively. Then the completions of uniform spaces (X,U_s) and (Y,V_s) through $(\beta_s X,\beta U_s)$ and $(\beta_s Y,\beta V_s)$, respectively, and the extension of the mapping $f:(X,U_s)\to(Y,V_s)$ on its completions $(\beta_s X,\beta U_s)$ and $(\beta_s Y,\beta V_s)$ are called Samuel compact extensions of uniform spaces (X,U) and (Y,V) respectively. If V_c is some precompact uniformity of the space Y contained in V, then the completion of the space (Y, V_c) is denoted by $(\mathrm{b}_c Y, \mathrm{b} V_c)$, and the extension of the mapping $f:(X,U_s)\to(Y,V_c)$ onto $(\beta_s X, \beta U_s)$ and $(b_c Y, bV_c)$ we denote as $\beta_c f$.

Theorem 3. *Let f :* $(X;\ U)\ \to\ (Y;\ V)$ *be a uniformly continuous mapping. Then the following conditions are equivalent:*

1. A mapping f is uniformly perfect.

2. A mapping f is precompact and for any compact extension $(b_c Y, bV_c)$ *of a uniform space* $(Y;\ V)$ *the mapping* $\beta_c f$ *satisfies to the condition* $\beta_c f\ (\beta_s X \setminus X) \subseteq b_c Y \setminus \mathrm{Y}$.

3. A mapping f is precompact and the mapping $\beta_s(\beta_s X \setminus \beta \mathrm{U}_s) \to (\beta_s \mathrm{Y} \setminus \beta V_s)$ *satisfies* $\beta_c f$ $(\beta_s X \setminus X) \subseteq b_c Y \setminus \mathrm{Y}$.

4. A mapping f is precompact and there is a compact extension $(b_c Y, bV_c)$ *of a uniform space* $(Y;\ V)$ *such that, for the extension* $\beta_s f:(\beta_s X \setminus \beta \mathrm{U}_s) \to (b_s \mathrm{Y} \setminus \mathrm{b} V_s)$ *of the mapping f the inclusion* $\beta_c f$ $(\beta_s X \setminus X) \subseteq b_c Y \setminus \mathrm{Y}$ *is fulfilled.*

Proof. Let us prove the implication 1 $\Rightarrow$ 2. Suppose that the mapping $f:\ (X;\ U)\ \to\ (Y;\ V)$ is uniformly perfect. Then, by definition of uniform perfectness the mapping f is precompact. Let us consider the extension $\beta_s f:(\beta_s X \setminus \beta \mathrm{U}_s) \to (b_s \mathrm{Y} \setminus \mathrm{bU}_s)$ of the mapping $f:(X, U_S) \to (Y, V)$. We put $Z = (\beta_c f)^{-1} Y$. Let $\tilde{f} = \beta_c f|_Z$. Then $\tilde{f}$ is a continuous extension onto Z of a perfect mapping f. But it is impossible to extend a perfect mapping $f:(X,U)\to(Y,V)$ onto any Hausdorff space Z containing X as a dense subset (see. Lemma 3.7.14 from [11]). So Z = X. Consequently $\beta_c f\ (\beta_s X \setminus X) \subseteq b_c Y \setminus \mathrm{Y}$.

The implications 2) $\Rightarrow$3) and 3) $\Rightarrow$ 4) are obvious. Implication 4) $\Rightarrow$1) follows from the following statement:

If g: Z $\to$ P is a perfect mapping, then for any $B \subset P$ the restriction $g_B : g^{-1}B \to B$ is perfect (see Prop. 3.7.4 from [11]).

Actually, if we set $Z = \beta_s X$ and $P = b_s Y, B = Y, g = \beta_c f$ then the condition 4) implies $(\beta_c f)^{-1} Y = X$ and that $f: X \to Y$ is a perfect mapping. Given that f is precompact, we conclude that the mapping $f:(X,U)\to(Y,V)$ is uniformly perfect. Theorem 3 is proved.

Note that if U is a precompact uniformity them any uniformly continuous mapping $f:(X,U)\to(Y,V)$ is precompact.

Taking this into account and assuming that U is a maximal precompact uniformity of a Tychonoff space X, then Theorem 3 implies well-known theorem of Henriksen and Isbell [12] in the form, contained in the book [10].

Corollary 3. . *For any continuous mapping* $f:(X,U)\to(Y,V)$ *the following conditions are equivalent:*

1) A mapping f is perfect.

2) For any compactication the extension $F_c : \beta X \to {}_c Y$ *of a mapping f satisfies* $F_\beta : (\beta X \setminus \mathrm{X}) \subseteq {}_c Y \setminus Y$.

3) Extension $F_\beta : \beta X \to \beta Y$ *of a mapping f satisfies* $F_\beta : (\beta X \setminus \mathrm{X}) \subseteq \beta Y \setminus Y$

4) There is a compactication ${}_c Y$ *, such that the extension* $F_c : \beta X \to {}_c Y$ *of a mapping f satisfies* $F_c : (\beta X \setminus \mathrm{X}) \subseteq {}_c Y \setminus Y$.

The set of all compactications of a uniformly continuous mapping $f:(X,U)\to(Y,V)$ will be denoted as $K(f)$. The set $K(f)$ is partially ordered by the order " $\le$ ", which we introduced earlier. A partially ordered set ($K(f)$; $\le$) is not empty (Theorem 1) and has a maximal element (Theorem 2).

We denote by $C(f)$ the set of all such uniformities U_P of a space X that, rstly UP U, and in the second a mapping $f:(X,U_S)\to(Y,V)$.is precompact and uniformly continuous. The set $C(f)$ is partially ordered by the inclusion " $\subseteq$ ". A partially ordered set ($C(f)$; $\subseteq$) is not empty and has a maximal element (Lemma 1).

Theorem 4. There is an isomorphism $G:(K(f),\le)\to(C(f),\subseteq)$ between the partially ordered sets $(K(f),\le)$ and $(C(f),\subseteq)$.

Proof. Suppose that $C(f)\in K(f)$, i.e. $cf:(cX,cU)\to(Y,V)$ is a uniformly perfect extension of the mapping $f:(X,U)\to(Y,V)$. Let Uc be a restriction of the uniformity cU on X. Then $U_c\subseteq U$ and the mapping $f:(X,U_S)\to(Y,V)$ is uniformly continuous and precompact. Consequently, $U_P\in P(f)$. Let $G(cf)=U_P$. It is easy to verify that the mapping $G:(K(f),\le)\to(C(f),\subseteq)$ is an isomorphism. Theorem 4 is proved.

Let U_m be a maximal uniformity of a space X and Y be a singleton. Then it turns $K(f)$ to be the set of all compact extensions of the space X and $C(f)$ becomes the set of all precompact uniformities $C(X)$ of the space X.

Then the partial order "$\le$ " will coincide with the natural order between compact extensions of the space X, and the inclusion " $\subseteq$ " will coincide with the natural order of "weak" and "strong" uniformity of the space X. Then the theorem implies well-known statement [11].

Corollary 4. *Posets (*$K(f)$*,*$\le$*) and (*$C(f)$*,*$\subseteq$*) are isomorphic.*

Let $f:(X,U)\to(Y,V)$ be a uniformly continuous mapping. The point $y\in Y$ is called the point of perfectness (see [5], [6]), if firstly $f^{-1}y$ is compact, in the second, for every neighborhood G of $f^{-1}y$ there exists a neighborhood O of $y\in Y$ such that $f^{-1}O\subseteq G$. The set of all points of perfectness of the mapping f we denote by $P(f)$ (see. [5], [6]).

Theorem 5. *Let* $f:(X,U)\to(Y,V)$ *be a uniformly continuous mapping,* $sf:(sX,sU)\to(Y,V)$ *be a Samuel compactification of the mapping* f.

Then $sf:(sX\setminus X)\subseteq Y\setminus \mathrm{P}(f)$

Proof. By construction $sf:(sX,\mathrm{sU})\to(Y,U)$ is a uniformly perfect mapping. Then the mapping $f_0=\left(f^{-1}P(f),U_{f^{-1}P(f)}\right)\to\left(P(f),\mathrm{V}_{P(f)}\right)$ is uniformly perfect, where $f_0=sf\big|_{f^{-1}P(f)}=f\big|_{f^{-1}P(f)}$ and $U_{f^{-1}P(f)}$, $V_{P(f)}$ are restrictions of the uniformities sU and V on $f^{-1}P(f)$ and $P(f)$ respectively. Then f_0 is uniformly perfect. Therefore, $f^{-1}P(f)$ is a closed subset in sX ([11]) and $f^{-1}P(f)\subseteq X$. $y\in sf(sX\setminus X)$.

Then there is a point $x\in sX\setminus X$ such that $sf(x) = y$ and $x\notin X$. From the perfectness of sf , and $[X]sX = sX$ it follows that $\left[f^{-1}y\right]_{sX}=sf^{-1}y$. But y can not belong to $P(f)$, as $f^{-1}y$ is complete and $f^{-1}y \subset f^{-1}P(f) \subset X$ with $f(x) = y$. This contradicts the fact that $x \notin X$. Consequently, $y \in Y \setminus P$. Theorem 5 is proved.

Remark 1. *If in Theorem 5 the mapping f is surjective, then the equality*

$$sf : (sX \setminus X) = Y \setminus \mathrm{P}(f) \qquad (*)$$

holds.

Theorem 6. *Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping. Then the inclusion $\beta_s f : (\beta X \setminus X) \setminus (\beta_s Y \setminus Y) \subseteq Y \setminus P(f)$ holds*.

Proof. Let $y \in \beta_s (\beta X \setminus X) \setminus (\beta_s Y \setminus Y)$.Then $y \notin \beta_s Y \setminus \mathrm{Y}$ and $y \in \mathrm{Y}$. There is a point $x \in \beta_s X \setminus X$ such that $\beta_s f(x) = y$, and $x \notin X$. Then $x \in f^{-1}P(f) = (\beta_s)^{-1} P(f)$. So $\beta_s f(x) = y \in Y \setminus P(f)$.

Remark 2. *If in Theorem 6 the mapping f is surjective, then thee quality*

$$\beta_s f (\beta_s X \setminus \mathrm{X}) \setminus (\beta_s \mathrm{Y} \setminus \mathrm{Y}) = Y \setminus P(f) \qquad (**)$$

holds.

A global research of points of closedness and perfectness of continuous mappings were carried out in [5], [6] and there were proved the equalities of type (*) and (**).

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